On the Groups of Isomorphisms of a System of Abelian Groups of Order p^m and Type $(n, 1, 1, \cdots, 1)$.*

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Introduction.

Early in the study of groups of isomorphisms Moore showed that the group of isomorphisms of an abelian group of order p^m and type $(1, 1, \cdots, 1)$ is the linear homogenous group,† extensively discussed by Jordan in his *Traité des Substitutiones* (1870). Miller discussed the automorphisms of an abelian group of order p^m, type $(m-1, 1)$,‡ and later gave incidentally a formula for the order of the group of isomorphisms of any abelian group of order p^m.§ In 1907 Rañum through his study of the group of classes of congruent matrices showed that the group of isomorphisms of any given abelian group of order p^m was simply isomorphic with a certain chief n-ary linear congruence group.|| In the present paper the viewpoint is different and the groups are treated as abstract groups. The object is to study the groups of isomorphisms of the system of abelian groups of ordem p^m, type $(n, 1, \cdots, 1), n > 1$, and to show that these groups of isomorphisms may be built upon the group of isomorphisms of an abelian group which contains no operators of order greater than p. To serve as a stepping stone to the general theory as well as to bring out the relations true for the first case, the case $n = 2$ will be considered immediately for $p = 2$ and for $p > 2$. In each development the group under consideration will be represented by G and its group of isomorphisms by I; p is used for an odd prime.

Theory.

Theorem 1. *The I of an abelian group of order 2^{m+1}, type $(2, 1, \cdots, 1)$ is of order $2^m(2^m-2)(2^m-2^2) \cdots (2^m-2^{m-1})$ and is simply isomorphic with a subgroup of index 2^m-1 in the holomorph of the abelian group of*

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† Cf. also Burnside, *Theory of Groups* (1897), §§ 171, 172 and Chap. XIV.

‡ Miller, *Transactions of the American Mathematical Society*, Vol. 2 (1901), pp. 259-264.

§ Miller, *Bulletin of the American Mathematical Society*, Vol. 20 (1913-14), p. 364.

|| Ranum, *Transactions of the American Mathematical Society*, Vol. 8 (1907), pp. 71-91.

order 2^m, type $(1, 1, \cdots, 1)$. This I may be obtained by extending an abelian group of order 2^m, type $(1, 1, \cdots, 1)$ by those operators from its own group of isomorphisms which leave one arbitrary operator in this abelian group fixed.

Suppose $m > 1$. The operators of order 2 in G evidently with the identity form a characteristic subgroup, H, of order 2^m. In H there is one (and only one) characteristic subgroup besides the identity. It is of order 2 and consists of the identity and the operator of order two which is the square of all the operators of order 4 in G. All the operators outside of H are of order 4. With the operators of H in identical correspondence any one of these operators may stand first, and an automorphism of G is then determined. Since these automorphisms are of order 2, commutative and number $2^m - 1$, the I of G contains an invariant abelian subgroup, H', of order 2^m, type $(1, 1, \cdots, 1)$. For, let $H \equiv 1, s_2, s_3, \cdots, s_h$, let $t^2 = s_2$, and let $G \equiv 1, s_2, s_3, \cdots, s_h, t, ts_2, \cdots, ts_h$ (all operators being commutative). Any operator ts_i, $i = 2, \cdots, h$, may correspond to t, so that the order of H' is h. Let $H' \equiv 1, v_2, v_3, \cdots, v_h$. Let the v that transforms t into ts_2 be v_2, into ts_3 be v_3, etc. Then, since each s is invariant under the v's

$$\begin{aligned} v_l^{-1} s_i v_l &= s_i, \\ v_l^{-1} t v_l &= t s_l. \end{aligned} \qquad l = 2, \cdots, h; \; i = 2, \cdots, h. \tag{1}$$

That the v's are of order 2 is evident from the fact that

$$v_l^{-1}(v_l^{-1} t v_l) v_l = v_l^{-1} t s_l v_l = v_l^{-1} t v_l \cdot v_l^{-1} s_l v_l = t s_l^2 = t;$$

and since all the operators of H' excepting the identity are of order 2, H' is abelian,* or,

$$\begin{aligned} v_a^{-1} v_b^{-1} t v_b v_a &= v_a^{-1} t s_b v_a = v_a^{-1} t v_a \cdot v_a^{-1} s_b v_a = t s_a s_b, \\ v_b^{-1} v_a^{-1} t v_a v_b &= v_b^{-1} t s_a v_b = v_b^{-1} t v_b \cdot v_b^{-1} s \; v_b = t s_b s_a, \end{aligned}$$

and since $ts_a s_b = ts_b s_a$, $v_a^{-1} v_b^{-1} t v_b v_a = v_b^{-1} v_a^{-1} t v_a v_b$, or $v_a v_b = v_b v_a$. That $v_a v_b$ transforms t into $ts_a s_b$ makes it possible to put H and H' into simple isomorphism in the following way: $s_i \sim v_i$, $i = 2, \cdots, h$.

From the nature of G, evidently H can be automorphic in all the ways an abelian group of order 2^m, type $(1, 1, \cdots, 1)$ can, except that s_2 must always correspond to itself. This means that one subgroup of order 2 in H is always fixed, so that the order of the quotient group of the I of G with respect to the invariant H' as a head is equal to the order of the group of isomorphisms of H divided by $2^m - 1$.*

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* Cf. Burnside, *loc. cit.*, p. 60.

* Burnside, *loc. cit.*, § 172.

It will now be shown that the I of G may be obtained by extending H' by operators which transform it in just the ways H may be transformed in G. This will be done by showing that an operator effecting any permissible automorphism of H, would produce a similar isomorphism among the operators of H'; i. e., if $u^{-1}s_h u = s_k$, then $u^{-1}v_h u = v_k$. It may be supposed that u is so chosen from the I of G that it transforms t into itself; for if not, u can be multiplied by such an operator from H' that the product will transform t into itself and at the same time effect exactly the same automorphism of H. Using (1),

$$(u^{-1}v_h u)^{-1}t(u^{-1}v_h u) = u^{-1}v_h^{-1}utu^{-1}v_h u = u^{-1}v_h^{-1}tv_h u = u^{-1}ts_h u = u^{-1}tu \cdot u^{-1}s_h u = ts_k, \text{just as } v_k^{-1}tv_k = ts_k; \qquad (2)$$

and since the v's are commutative with the s's and so also is $u^{-1}v_l u$ (because $(u^{-1}v_l u)^{-1}s_i(u^{-1}v_l u) = u^{-1}v_l^{-1}us_i u^{-1}v_l u = s_i$, for $us_i u^{-1}$ is some s, and hence $v_l^{-1}us_i u^{-1}v_l = us_i u^{-1}$, and $u_l^{-1}us_i u^{-1}u = s_i$); therefore, $u^{-1}v_h u$ and v_k effect the same isomorphisms of G with itself. Thus, $u^{-1}v_h u = v_k$.

From the preceding it is obvious that the I of G is a subgroup of index $2^m - 1$ in the holomorph of the abelian group of order 2^m, type $(1, 1, \cdots, 1)$. This I should have exactly $\phi(4)$, or 2 invariant operators.* The operator besides the identity is easily shown to be v_2 according to the notation here used. (Note too that here v_2 corresponds to s_2 in an invariant subgroup of index 2^{m-1} in H). All the v's are commutative with v_2, and if in (2) $h = 2$ remembering that $u^{-1}s_2 u = s_2$), the result from the end of the preceding paragraph is $u^{-1}v_2 u = v_2$.

Next, the case in which p is an odd prime will be considered, and it will be shown that in this case the I of G is a direct product of two groups; and what these two groups are will be discussed.

The operators of order p in G evidently with the identity form a characteristic subgroup, J, of order p^m; also in J there is a characteristic subgroup, H, of order p whose $p - 1$ operators of order p are the pth powers of the operators of order p^2 in G. The operators of order p^2 correspond among themselves in every automorphism of G. With J in identical correspondence, any one of the p^m operators of order p^2 having the same pth power in H may stand first, and the automorphism of G is then fixed. Moreover, every such automorphism of G is of order p, and these $p^m - 1$ automorphisms of order p are commutative. These two facts may be proved just as similar facts were proved in the preceding case where the prime was 2. Let this invariant abelian subgroup of order p^m and type $(1, 1, \cdots, 1)$ in the I of G be E, and let its oper-

* Miller, Blichfeldt, and Dickson, *Finite Groups* (1916), p. 162.

ators be v's. If the generators of J are $s_1, s_2, \cdots, s_m$ where s_1 generates H (and $t^p = s_1$), then by a method analogous to that used for the even prime 2, it can be shown that the correspondence between J and E can be taken as $s_i \sim v_i$ $(i = 1, \cdots, m)$, where $v_i^{-1} t v_i = t s_i$ and where $v_i^{-1} s_l v_i = s_l$ $(i = 1, \cdots, m;\ l = 1, \cdots, m)$.

From the nature of G, evidently J can be automorphic in all the ways an abelian group of order p^m, type $(1, 1, \cdots, 1)$ can, excepting that H must always correspond to itself. Since J has $(p^m - 1)/(p - 1)$ subgroups of order p,* this means that the order of the quotient group of the I of G with respect to the invariant E is equal to the order of the group of isomorphisms of J divided by $(p^m - 1)/(p - 1)$, which gives the order of the I of G as $p^m(p-1)(p^m - p)(p^m - p^2) \cdots\cdots (p^m - p^{m-1})$.

Consider those automorphisms of G in which the operators of H are in identical correspondence. Suppose that the operator, u, effecting the isomorphism under consideration transforms t into itself; for if not, u can be multiplied by such an operator from E that the product transforms t into itself and at the same time effects exactly the same automorphism of J. As in the preceding theorem, it can be shown easily that u transforms the operators of E among themselves in the same way it transforms the corresponding operators of J; that is, if

$$u^{-1} s_1 u = s_1, \quad u^{-1} s_j u = s_{j'}, \quad (j = 2, \cdots, m;\ j' = 2, \cdots, m);$$

then $u^{-1} v_1 u = v_1$, $u^{-1} v_j u = v_{j'}$, (j and j' have the same values respectively before).

Since all the automorphisms of G with the operators of the characteristic subgroup H in identical correspondence have been considered, the I of G evidently contains an invariant subgroup I' which is simply isomorphic with an abelian group of order p^m, type $(1, 1, \cdots, 1)$ extended by those operators from its own group of isomorphisms that leave the operators of one and only one of its subgroups of order p fixed. Since all other automorphisms of G arise from the automorphisms of H, and H is cyclic and of order p, obviously the quotient group of I with respect to I' is a cyclic group of order $p - 1$. It will now be shown that the I of G is simply isomorphic with the direct product of I' and the cyclic group of order $p - 1$.

The central of the I of G is the group of totitives (mod p^2) of order $\phi(p^2) = p(p-1)$; that is, a cyclic group, the product of a cyclic group of order p by a cyclic group of order $p - 1$. Each of these operators in the central of the I of G transforms every operator of G into the same power of

* Burnside, *loc. cit.*, p. 59.

itself.* The cyclic group of order p in this central has been already obtained. In the notation here used, it is generated by v_1, since $v_1^{-1}tv_1 = ts_1 = t^{1+p}$, $v_1^{-2}tv_1^2 = ts_1^2 = t^{1+2p}$, etc.; and since, for any s, $v_1^{-1}s_iv_1 = s_i$ $(i = 1, \cdots, m)$, and the $(1 + kp)$th power of s_i is s_i; and since $u^{-1}v_1u = v_1$. If now $s_1 \sim s_1^n$, then since $t^p = s_1$, t can $\sim t^n$, and the remainder of the automorphism of G may be set up by having the other generators (besides s_1) of J correspond to their own nth powers, $s_i \sim s_i^n$ $(i = 2, \cdots, m)$. If w effects this automorphism of G, $w^{-1}s_iw = s_i^n$ $(i = 1, \cdots, m)$, $w^{-1}tw = t^n$ (also $w^{-1}s_i^aw = s_i$, $w^{-1}t^aw = t$), it is necessary and sufficient to show that w is commutative with all the v's and with u. First, it will be shown that $w^{-1}v_iw = v_i$. Since $v_i^{-1}tv_i = ts_i$ $(i = 1, \cdots, m)$, $(w^{-1}v_iw)^{-1}t(w^{-1}v_iw) = w^{-1}v_i^{-1} \cdot wtw^{-1} \cdot v_iw$ $= w^{-1} \cdot v_i^{-1}t^av_i \cdot w = w^{-1}(v_i^{-1}tv_i)^aw = w^{-1}(ts_i)^aw = w^{-1}t^aw \cdot w^{-1}s_i^aw = ts_i$, just as $v_i^{-1}tv_i = ts_i$, and since the v's are commutative with s_j $(j = 1, \cdots, m)$ and so also is $w^{-1}v_iw$ (because $(w^{-1}v_iw)^{-1}s_j(w^{-1}v_iw) = w^{-1}v_i^{-1} \cdot ws_jw^{-1} \cdot v_iw$ $= w^{-1}v_i^{-1}s_j^av_iw = w^{-1}s_j^aw = s_j$); therefore, these two operators from I are identical or $w^{-1}v_iw = v_i$. Second, to show that w is commutative with u, use will be made of

$$\left\{\begin{array}{l} u^{-1}s_1u = s_1, \\ u^{-1}s_ju = s_{j'} \end{array}\right. \quad (j = 2, \cdots, m;\ j' = 2, \cdots, m) \text{ and } u^{-1}tu = t. \text{ Here}$$

$$(w^{-1}uw)^{-1}s_j(w^{-1}uw) = w^{-1}u^{-1} \cdot ws_jw^{-1} \cdot uw = w^{-1}u^{-1}s_j^auw \quad (j = 1, \cdots, m)$$

$$= \left\{\begin{array}{l} \text{if } j = 1,\ w^{-1}s_1^aw = s_1, \text{ just as } u^{-1}s_1u = s_1. \\ \text{if } j = 2, \cdots, m,\ w^{-1}s_{j'}^aw = s_{j'}, \text{ just as } u^{-1}s_ju = s_{j'}; \end{array}\right.$$

also $(w^{-1}uw)^{-1}t(w^{-1}uw) = w^{-1}u^{-1} \cdot wtw^{-1} \cdot uw = w^{-1}u^{-1}t^auw = w^{-1}t^aw = t$.

Hence, not only is the quotient group of I with respect to I' the cyclic group of order $p - 1$, but I contains such a cyclic group whose operators (excepting the identity) lie outside of I' and are commutative with each of the operators of I'. Therefore, I is simply isomorphic with the direct product of I' and the cyclic group of order $p - 1$, and for $m > 1$ there results the

Theorem 2. *The I of an abelian group of order p^{m+1}, type $(2, 1, \cdots, 1)$ is of order p^m $(p - 1)(p^m - p) \cdots (p^m - p^{m-1})$ and is simply isomorphic with the direct product of a cyclic group of order $p - 1$ and the group formed by extending an abelian group of order p^m, type $(1, 1, \cdots, 1)$ by all those operators from its own group of isomorphisms which leave the operators of some one of its subgroups of order p in identical correspondence.*

As a side step from the main problem of this paper the following general

* Miller, *Transactions of the American Mathematical Society*, Vol. 2 (1901), p. 260.

proposition concerning a property obtaining in the abelian group just considered, will now be discussed.

Subsidiary Theorem. *If an abelian group G contains a characteristic subgroup H of prime order p, the I of G is simply isomorphic with a direct product, one factor of which is the cyclic group of order $p-1$. This I is then divisible if $p > 2$.*

Since "every abelian group is the direct product of its Sylow subgroups whenever its order is the product of more than one different prime," * H is in the Sylow subgroup, S, of order p^m and type $(m_1, m_2, \cdots, m_l)$, $m_1 \geqq m_2 \geqq \cdots \geqq m_l$, where $m = m_1 + m_2 + \cdots + m_l$ and $m_1 > 0$. If m_1 is not greater than m_2, there is no characteristic subgroup of order p. If $m_1 > m_2$, the group of order p in the cyclic group of order p^{m_1} is a characteristic subgroup of G (and the only characteristic subgroup of order p), since its operators are the p^{m_1-1}th powers of all the operators of S, its operators of order p being the p^{m_1-1}th powers of the operators of order p^{m_1} in S. Incidentally then, it has been shown that *a necessary and sufficient condition that an abelian group of order p^m contain a characteristic subgroup of order p is that there be one and only one largest invariant.* This group, H, of order p is a fundamental characteristic subgroup.† The remainder of the proof will now be worked out with respect to S, since the I of G is the direct product of the groups of isomorphisms of its Sylow subgroups.

The operators effecting the automorphisms of S in which the operators of H remain in identical correspondence form an invariant subgroup, I', of the group of isomorphisms of S (I_S). The quotient group of I_S with respect to I' is the group of isomorphisms of H; i. e., the cyclic group of order $p-1$. In the notation here used and with $p > 2$, the central of I_S is a cyclic group of order $\phi(p^{m_1}) = p^{m_1-1}(p-1)$, the product of a cyclic group of order p^{m_1-1} and another cyclic group of order $p-1$, and each of these operators in the central of I_S transforms every operator of S into the same power of itself.‡ The cyclic group of order p^{m_1-1} is in I', its operators being those which transform operators of order p^{m_1} in S into their $(1+kp)$th powers, $k = 1, 2, \cdots, p^{m_1-1}$. The operators of order p in H are invariant individually under such transformations, since these powers of each opeartor of order p in H are that operator itself. The operator of I_S which transform every operator of S into the 2nd, 3d, $\cdots$, $(p-1)$th powers of itself, evidently transform

* Miller, Blichfeldt, and Dickson, *loc. cit.*, p. 87.

† Cf. Miller, AMERICAN JOURNAL OF MATHEMATICS, Vol. XXVII (1905), p. 15; also Miller, Blichfeldt, and Dickson, *loc. cit.*, p. 110.

‡ Miller, *Transactions of the American Mathematical Society*, Vol. 1 (1900), p. 397; Vol. 2 (1901), p. 260.

the operators of H among themselves, so are not in I'. They with the identity, constitute a cyclic group of order $p-1$, and since they are in the central of I_S; they are individually commutative with the operators of I'. Hence, since the quotient group of I_S with respect to I' is a cyclic group of order $p-1$, and I_S contains a cyclic group of order $p-1$ having only the identity in common with I' and each of its operators is commutative with each of the operators of I', therefore I_S is simply isomorphic with the direct product of I' and a cyclic group of order $p-1$.

The theory of the second theorem will now be extended to any abelian group, G, of order p^{m+n-1}, type $(n, 1, 1, \cdots (m-1)$ units), $n > 1$, p a prime > 2. Let t be the operator of order p^n, and $s_2, \cdots, s_m$ independent generators of order p, and for convenience let $t^{p^{n-1}} = s_1$. From the observations under the Subsidiary Theorem, s_1 generates a characteristic cyclic subgroup, $H^{(n)}$, of order p. All the operators of order p in G form a characteristic abelian subgroup $H^{(n-1)}$ of order p^m, type $(1, \cdots, 1)$. $H^{(n)}$ is in $H^{(n-1)}$. Then there is a characteristic abelian subgroup, $H^{(n-2)}$, of order p^{m+1}, type $(2, 1, \cdots, 1)$ generated by the $p^{(n-2)}$th powers of operators of order p^n and $H^{(n-1)}$. So likewise, $H^{(n-3)}$ is generated by the p^{n-3}th powers of operators of order p^n and $H^{(n-1)}$; and in general, $H^{(n-r)}$, of order p^{m+r-1} and type $(r, 1, \cdots, 1)$, is generated by the p^{n-r}th powers of the operators of order p^n and $H^{(n-1)}$, $r = 2, \cdots, n-1$. Each of these characteristic subgroups contains the preceding, and the largest $(H^{(1)})$ is of index p in G itself. This series of subgroups forms a characteristic series of G.*

Now with the operators of $H^{(1)}$ in identical correspondence, evidently t can correspond to any one of $p^m - 1$ operators besides itself; it can correspond to itself multiplied by any one of the operators of $H^{(n-1)}$, since these products (and t) alone are of order p^n and have the same pth power that t has in $H^{(1)}$. As was shown in connection with Theorem 2, these isomorphisms (excepting the identity) are of order p and commutative. Hence, the I of G contains an invariant abelian subgroup, E, of order p^m, type $(1, \cdots, 1)$, which is simply isomorphic with $H^{(n-1)}$; and, moreover, if the same convenient notation be employed here as in the theorem to which reference has just been made, the correspondence, $v_i \sim s_i$, $i = 1, \cdots, m$, can be set up, where the v's are the independent generators of the subgroup E. The transformations, accordingly, are $v_i^{-1} t v_i = t s_i$, $(i = 1, \cdots, m)$, $v_i^{-1} s_j v_i = s_j$, $(i = 1, \cdots, m; j = 1, \cdots, m)$.

Now, let the operators of $H^{(2)}$ be in identical correspondence. t^p can

* Frobenius, *Berliner Sitzungsberichte* (1895), p. 1027; cf. Burnside, *loc. cit.*, §§ 163, 164.

correspond to $t^{p^{n-1}+p}$, $t^{2p^{n-1}+p}$, $\cdots$, $t^{(p-1)p^{n-1}+p}$ (or $t^p s_1$, $t^p s_1^2$, $\cdots$, $t^p s_1^{p-1}$, respectively, since $t^{p^{n-1}} = s_1$) besides itself, because these and these alone have the same pth power in $H^{(2)}$ and at the same time are themselves the pth powers of operators outside $H^{(1)}$. Simultaneously with $t^p \sim t^{p^{n-1}+p}$, t must correspond to some operator of order p^n whose pth power is $t^{p^{n-1}+p}$; such operators are $t^{p^{n-2}+1}$ times operators from $H^{(n-1)}$. The operators from $H^{(n-1)}$ may be supposed to be in $H^{(n)}$ also, for if not, the operator effecting the automorphism of G under consideration can be multiplied by such a v that $H^{(1)}$ is transformed as stated in the preceding and t corresponds to $t^{p^{n-2}+1}$ times some operator from $H^{(n)}$. Accordingly, this isomorphism of G may be said to be effected by an operator which transforms every operator into its $(kp^{n-2}+1)$th powers, $k = 1, \cdots, p-1$.

Similarly, if the operators of $H^{(3)}$ are in identical correspondence, the additional isomorphisms of G spring from those in which the operators correspond to their $(kp^{n-3}+1)$th powers, $k = 1, \cdots, p-1$.

More generally, if the operators of $H^{(r)}$, $r = 2, \cdots, n-1$, are in identical isomorphism, $t^{p^{r-1}}$ can correspond to $t^{p^{n-1}+p^{r-1}}$, $t^{2p^{n-1}+p^{r-1}}$, $\ldots$, $t^{(p-1)p^{n-1}+p^{r-1}}$ (or $t^{p^{r-1}}s_1$, $\cdots$, $t^{p^{r-1}}s_1^{p-1}$, respectively) besides itself, because these operators and these only have the same pth power in $H^{(r)}$ and are themselves the p^{r-1}th powers of operators outside $H^{(1)}$. These isomorphisms are those effected by an operator which transforms the operators of G into their $(kp^{n-r}+1)$th powers, $k = 1, \cdots, p-1$. These automorphisms are p^{n-2} in number (because $r = 2, \cdots, n-1$ and $k = 1, \cdots, p-1$). If when $r = 1$, v_1 is included, these isomorphisms number p^{n-1}, and they are, moreover, those in which the operators of G go over into their $(1+kp)$th powers, $k = 1, 2, \cdots, p^{n-1}$. The only other powers are the 1st, 2nd, $\cdots$, $(p-1)$th, and when these are effected the characteristic subgroup $H^{(n)}$ takes all its automorphisms. But the operators effecting the possible transformations of all the operators into their same powers constitute the central of the I of G, a cyclic group of order $\phi(p^n) = p^{n-1}(p-1)$, (because the highest order of operators in G is p^n), the product of two cyclic groups, one of order $p-1$ and one of order p^{n-1}. From the Subsidiary Theorem the I of G is the direct product of this cyclic group of order $p-1$ and another subgroup, I'. The cyclic group of order p^{n-1} must be in I'. Suppose u to be employed to represent a generator of this cyclic group, F, of order p^{n-1}, so that $u^{-1}tu = t^{1+kp}$, $k = 1, \cdots, p^{n-1}$. With the operators of $H^{(n-1)}$ in identical correspondence, all the possible isomorphisms of G are effected by u and the v's. They all are commutative (since u is in the central of I) and the cross-cut of E and F is the cyclic group of order p generated by $v_1 (= u^{p-2})$.

Hence, I' contains an invariant abelian subgroup of order p^{m+n-2}, type $(n-1, 1, 1, \cdots, 1)$.

Finally, if the operators of $H^{(n)}$ alone are in identical correspondence, the remaining operators of $H^{(n-1)}$ have exactly the automorphisms of an abelian group of order p^{m-1}, type $(1, 1, \cdots, 1)$ when the operators of some one subgroup of order p remain fixed. If w effects such an automorphism (and leaves t invariant), it can be shown just as in Theorem 2 that w is commutative with v_1 but transforms the other operators of E just as the operators of $H^{(n-1)}$ outside of $H^{(n)}$ are transformed, and w, furthermore, would be found to be commutative with u. The number of the isomorphisms effected by w's would be $(p^m - p)(p^m - p^2) \cdot\cdot\cdot\cdot\cdot (p^m - p^{m-1})$.*

The following may be stated as a summary of these results:

Theorem 3. *The I of an abelian group G of order p^{m+n-1}, type $(n, 1, 1, \cdots, 1)$, p a prime > 2, $n > 2$, is of order $(p-1)p^{m+n-2}(p^m - p)(p^m - p^2) \cdot\cdot\cdot\cdot\cdot (p^m - p^{m-1})$, and is simply isomorphic with the direct product of a cyclic group of order $p-1$ and a group formed by extending an abelian group of order p^m, type $(1, 1, \cdots, 1)$ by all those operators from its own group of isomorphisms which leave invariant the operators of one its own group of isomorphisms which leave invariant the operators of one cyclic group of order p, and then multiplying this extended group by an operator of order p^{n-1} which is commutative with each operator of the extended group and which has one of the invariant operators of order p for its p^{n-2}th power.*

This shows that *for a given odd prime p and a fixed value of $m > 0$, the group of isomorphisms of each abelian group of the system $(n = (2), 3, 4, \cdots,)$ contains the group of isomorphisms of the preceding as an invariant subgroup of index p,* since they differ only in the order of the operator by which the extended group is multiplied.

Again, since multiplying the extended group by the designated operator of order p^{n-1} is equivalent to taking an abelian group of order p^{m+n-2}, type $(n-1, 1, \cdots, 1)$ and extending it by all those operators from its own group of isomorphisms that leave invariant the operators of exactly one of its subgroup of order p^{n-1}, the preceding theorem can be stated as follows, and from it can be seen that *the group of isomorphisms of each abelian group of the system under study is an extension of the one of the system just before it and of index p in it.*

Theorem 3′. *The I of an abelian group G of order p^{m+n-1}, type $(n, 1, 1, \cdots, 1)$, p an odd prime and $n > 1$, is of order $(p-1)p^{m+n-2}(p^m - p)$*

* Cf. Burnside, *loc. cit.*, § 48.

$(p^m - p^2) \cdots (p^m - p^{m-1})$ and is simply isomorphic with the direct product of a cyclic group of order $p - 1$ and the group formed by extending an abelian group of order p^{m+n-2}, type $(n - 1, 1, \cdots, 1)$ by all those operators from its own group of isomorphisms that leave invariant the operators of exactly one of its cyclic subgroups of order p^{n-1}.

If p is the even prime and $n > 2$, F is not cyclic but is an abelian group of order 2^{n-1}, type $(n - 2, 1)$,* and the 2^{n-3}th power of the operators of order 2^{n-2} generates the cross-cut of F and E, a group of order two. Accordingly, the counterparts of the preceding two theorems are:

Theorem 4. *The I of an abelian group G of order 2^{m+n-1}, type $(n, 1, 1, \cdots, 1)$, $n > 2$, is of order $2^{m+n-2}(2^m - 2)(2^m - 2^2) \cdots (2^m - 2^{m-1})$ and is simply isomorphich with the direct product of a group of order 2 and a group formed by extending an abelian group of order 2^m, type $(1, 1, \cdots, 1)$ by all those operators from its own group of isomorphisms which leave invariant one operator of order two, and then multiplying this extended group by an operator of order 2^{n-2} which is commutative with each operator of the extended group and which has the invariant operator of order two for its 2^{n-3}th power.*

This and the following equivalent statement of the proposition show the inclusive relation between the groups of isomorphisms of two consecutive groups of the system:

Theorem 4. *The I of an abelian group of order 2^{m+n-1}, type $(n, 1, 1, \cdots, 1)$, $n > 2$, is of order $2^{m+n-2}(2^m - 2)(2^m - 2^2) \cdots (2^m - 2^{m-1})$ and is simply isomorphic with the direct product of a group of order 2 and the group formed by extending an abelian group of order 2^{m+n-3}, type $(n - 2, 1, \cdots, 1)$ by all those operators from its own group of isomorphisms that leave invariant individually the operators of exactly one of its cyclic subgroups of order 2^{n-2}.*

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* Burnside, *loc. cit.*, § 169.